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THE STABILITY OF THE SET OF EQUILIBRIUM POSITIONS OF AUTONOMOUS MECHANICAL SYSTEMS WITH SLIDING FRICTION†

V. M. MATROSOV and I. A. FINOGENKO

Moscow

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Problems of the stability of the set of equilibrium positions of autonomous differential equations, to which the equations of motion of mechanical systems with sliding friction can be reduced, are considered (see [1, 2]). The structure of the equations, due to the specific features of the system, is taken into account substantially and the general properties of motions, investigated previously in [3, 7], and the sets which arise when analysing the equations and which possess the properties of the absolute sector are used.

1. FORMULATION OF THE PROBLEM

The equations of motion of the mechanical system in question for generalized coordinates $q = (q^1, \ldots, q^k)$ (written in vector form) are

$$A(q)\ddot{q} = q(q,\dot{q}) + Q^A(q,\dot{q}) + Q^T(q,\dot{q},\ddot{q}) \tag{1.1}$$

Here $A(q) = [a_{ij}(q)]_1^k$ is the matrix of the inertia coefficients, $Q^A(q, \dot{q})$ and $g(q, \dot{q})$ are vector functions, representing the active forces, generalized gyroscopic forces, translational inertial forces and other forces, and $Q^T(q, \dot{q}, \ddot{q})$ are generalized friction forces, expressed by the formulae

$$Q_s^T(q,\dot{q},\ddot{q}) = \begin{cases} -f_s(q^s,\dot{q}^s)|N_s|\operatorname{sgn}\dot{q}^s, & \text{if } \dot{q}^s \neq 0 \\ f_s(q^s,0)|N_s|\operatorname{sgn}Q_s^{T0}, & \text{if } \dot{q}^s = 0; \quad |Q_s^{T0}| > f_s(q^s,0)|N_s|_{\ddot{q}^s=0} \\ Q_s^{T0}, & \text{if } \dot{q}^s = 0; \quad |Q_s^{T0}| \leq f_s(q^s,0)|N_s|_{\ddot{q}^s=0} \end{cases}$$

$$Q_s^{T0} \triangleq \sum_{j=1,\ j\neq s}^{k} a_{s,j}(q)\ddot{q}^j - [g_s(q,\dot{q}) + Q_s^A(q,\dot{q})]_{\ddot{q}^s=0} \tag{1.2}$$

$N_s = N_s(t, q, \dot{q})$, $1 \leq s \leq k_*$, $k_* \leq k$, where $f_s(q^s, \dot{q}^s) > 0$ are the coefficients of friction, $|N_s|$ are the moduli of the normal reactions, and Q_s^{T0} are the friction forces in the case of relative rest; we assume $f_s = 0$ for $s = k_* + 1, \ldots, k$.

The equations of motion (in the general non-autonomous case) are described in more detail in [1, 2] and were investigated in [3–7], where the conditions for Eqs (1.1) to be solvable for $\ddot{q}$ were obtained and they were reduced to the form

$$\ddot{q} = G(q,\dot{q}) \tag{1.3}$$

with a, generally speaking, discontinuous function $G: \Omega \to R^k$, defined in a certain region $\Omega \subset R^{2k}$, the existence of right-sided solutions of (1.3) was proved and their general properties were investigated. These conditions (inequalities (3.1) in [2] and inequalities (1.2) in [7]) and also the necessary properties of continuity and differentiability of the functions occurring in (1.1) and (1.2) are henceforth assumed to be satisfied.

When using Lyapunov's functions to analyse implicit systems of the form (1.1), difficulties may arise when establishing that the derivatives of these functions are sign-definite, by virtue of Eq. (1.1). Here we investigate the properties of the solutions of Eq. (1.1) in the region of the set of equilibrium positions, which enable us to reduce these difficulties. The solutions are thought of as being right-hand sided. To simplify the later notation and proofs, Eq. (1.3) is converted to the form

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$$\dot{x} = f(x); \quad x = (q, \dot{q}), \quad f = (G_1, G), \quad G_1 = \dot{q} \tag{1.4}$$

For each point $x \in \Omega$ we put

$$\Gamma = \Gamma(x) \triangleq \{x' = (q', \dot{q}') \in R^{2k} : \dot{q}'^{s'} = 0, \text{ if } s \in \mathcal{N}, \ f_s \mid N_s \mid > \mid Q_s^{T0} \mid;$$
$$\dot{q}'^s Q_s^{T0} \leqslant 0, \text{ if } s \in \mathcal{N}, \ f_s \mid N_s \mid \leqslant \mid Q_s^{T0} \mid, \ \mid N_s \mid \neq 0\}$$
$$\mathcal{N} = \mathcal{N}(\dot{q}) \triangleq \{s \in (1, \ldots, k_*) : \dot{q}^s = 0\}$$

where the value of the functions f_s, $|N_s|$, Q_s^{T0} correspond to $(q, \dot{q})$, and the quantity $\ddot{q}$ in the expressions for $|N_s|$ and Q_s^{T0} is assumed to be equal to $G(q, \dot{q})$.

We put

$$S_\delta = S_\delta(x) \triangleq \{x' \in \Omega : \| x - x' \| < \delta\}, \quad \Omega_\delta = \Omega_\delta(x) \triangleq S_\delta(x) \cap \Gamma(x)$$

If $\mathcal{N} = 0$ or $|N_s| = 0$ for all $s \in \mathcal{N}$, we assume $\Gamma \triangleq R^{2k}$.

We will present some properties of the function f, which will be required later.

Lemma 1.1. The following properties hold at each point $x \in \Omega$

1. the function f is locally bounded;
2. the function f is continuous along the set $\Gamma(x)$, i.e. for any $\varepsilon > 0$ a $\delta = \delta(\varepsilon, x) > 0$ exists such that $\| f(x) - f(x') \| < \varepsilon \forall x' \in \Omega_\delta(x)$;
3. $f(x) \in \Gamma(x)$;
4. a locally Lipschitz function V_x exists: $R^{2k} \to R^1$ such that

$$V_x(x') \geqslant 0; \quad V_x(x') = 0 \Leftrightarrow x' \in \Gamma(x) \tag{1.5}$$

and, if $\Gamma(x) \neq R^{2k}$, numbers $\alpha = \alpha(x) > 0$ and $\delta = \delta(x) > 0$ exist such that

$$D^+ V_x(x') < -\alpha \forall x' \in S_\delta(x) \backslash \Gamma(x) \tag{1.6}$$

where $D^+ V_x(x') \triangleq \lim_{h \to +0} [V_x(x' + hf(x')) - V(x')] / h$ is the right derivative, by virtue of Eq. (1.4).

Properties 1 and 2 (in a different notation) were established in [5] (Lemmas 1 and 4). Property 3 follows from (1.2), Lemma 2 [2] and the definition of the sets Γ. Property 4 was proved in [5] (Lemma 8) as it applies to the Δ-solutions of Eq. (1.3), i.e .they are continuous functions, which (in the case of autonomous equation (1.3)) satisfy the relations

$$D^+ q(t) = \dot{q}(t), \quad \| D^+ \dot{q}(t) - G(q(t), \dot{q}(t)) \| < \Delta, \quad \forall t \in [0, a)$$

Here it is only necessary to note that any right-hand sided solution of Eq. (1.4) is also a Δ-solution of Eq. (1.3), while the right (upper right) derivative of a locally Lipschitz function along the right-hand sided solution can be evaluated using Yoshizawa's theorem [8].

2. ADDITIONAL ASSERTIONS

It follows from (1.5) and (1.6) that for any $x \in \Omega$, by an appropriate choice of the number $\delta > 0$, the set $\Omega_\delta(x)$ possesses the property of the absolute sector, namely, the trajectory of any solution with initial condition $x(0) \in \Omega_\delta(x)$ remains within $\Omega_\delta(x)$ for all $t \geqslant 0$ for which $x(t) \in S_\delta(x)$. In order to indicate this property of the set $\Omega_\delta(x)$ we will call it the absolute sector, generated by the set $\Gamma(x)$ or, more briefly, the Γ-sector with vertex x and radius δ, which always has a form such that for a number $\delta > 0$ condition (1.6) is satisfied. For the case when $\Gamma(x) = R^{2k}$, the equation $\Omega_\delta(x) = S_\delta(x)$ is satisfied, and the radius is assumed to be an arbitrary positive number. If $\Gamma(x) \neq R^{2k}$ and $\delta > 0$ is the radius of the Γ-sector, then any number $\delta' \in (0, \delta)$ is also the radius of the sector, generated by the same set $\Gamma(x)$. This obvious property will later be used without reservations. In particular, we will always assume that for any $x \in \Omega$ the choice of δ ensures that the condition $\overline{S_\delta(x)} \subset \Omega$ is satisfied, where the bar denotes closure of the set. It then follows from Theorem 1 [6] on continuability, that any solution $x(t)$, defined in the right maximum interval $[0, \omega)$ with initial condition $x(0) \in S_\delta(x)$, exists for all $t \geqslant 0$ for which $x(t) \in \overline{S_\delta(x)}$.

For an arbitrary set $M \subset R^{2k}$ and number $\beta > 0$ we will denote the β-neighbourhood of the set M by M^β, i.e. $M^\beta \triangleq \{x' \in R^{2k} : d(x', A) < \beta\}$, where d is the distance from the point to the set. For each $x \in \Omega$, we will denote the β-neighbourhood of the set $\Gamma(x)$ by $\Gamma^\beta(x)$.

Lemma 2.1. Suppose $x \in \Omega$ and $\delta = \delta(x) > 0$ is the radius of the β-sector $\Omega_\delta(x)$. Then, for any numbers $\tau > 0$, $\beta \in (0, \delta)$ values $\tau_0 \in (0, \tau)$, $\beta_0 \in (0, \beta)$ exist such that for any solution $x(t)$

$$(x(0) \in \Gamma^{\beta_0}(x) \cap S_{\delta-\beta}(x)) \Rightarrow (x(\tau_0) \in \Gamma(x) \cap S_\beta(x(0))) \tag{2.1}$$

Proof. Suppose $\beta \in (0, \delta)$ and $\tau > 0$ are arbitrary. By virtue of the local boundedness, the function f is bounded in a compact set $S_\delta(x)$. Hence $\tau_0 \in (0, \tau)$ exists and is so small that for all solutions $x(t)$

$$(x(0) \in S_{\delta-\beta}(x)) \Rightarrow (\forall t \in [0, \tau_0], \quad x(t) \in S_\beta(x(0))) \tag{2.2}$$

If $\Gamma(x) = R^{2k}$, (2.1) follows from (2.2) for arbitrary $\beta_0 > 0$.

Suppose $\Gamma(x) \neq R^{2k}$ and $\alpha = \alpha(x) > 0$ is a number for which inequality (1.6) holds. Using condition (1.6) we choose $\beta_0 \in (0, \beta)$ so that

$$V_x(x') < \alpha\tau_0 \forall x' \in \Gamma^{\beta_0}(x) \cap S_\delta(x) \tag{2.3}$$

We will assume that $x(t)$ is the solution with initial condition $x(0) \in \Gamma^{\beta 0}(x) \cap S_{\delta-\beta}(x)$. It then follows from (2.2) that

$$x(t) \in S_\beta(x(0)), \quad \forall t \in [0, \tau_0] \tag{2.4}$$

If $x(0) \in \Gamma(x)$ here, we conclude from (2.4) and the obvious relation $S_\beta(x)) \subset S_\delta(x)$ and $x(t)\ \Omega_\delta(x)$, $\forall t \in [0, \tau_0]$, and inclusion (2.1) is proved. If $x(0) \notin \Gamma(x)$, we obtain from (1.6) and (2.3)

$$V_x(x(t)) < V_x(x(0)) - \alpha t < \alpha(\tau_0 - t)$$

for all $t > 0$, such that $x(t) \in S_\delta(x) \backslash \Gamma(x)$. Since the function V_x is non-negative and the inclusion $x(t) \in S_\delta(x)$ is satisfied in the section $[0, \tau_0]$, we obtain a point $t_1 \in (0, \tau_0)$ such that $V_x(x(t_1)) = 0$. Then $x(t_1) \in \Gamma(x)$ and $x(t) \in \Omega_\delta(x)$ for all $t \in [t_1, \tau_0]$, whence, taking (2.4) into account, we obtain (2.1).

Everywhere henceforth we will denote by $M \subset \Omega$ the compact set which satisfies the condition $x \in M$ for each $M \subset \Gamma(x)$. The last inclusion is obviously always satisfied if M is a set of equilibrium positions of Eq. (1.1).

Lemma 2.2. For any $\varepsilon > 0$ a $\varepsilon_0 \in (0, \varepsilon)$, a finite set of Γ-sectors $\Omega_{\delta_i}(x_i)$ with vertices at the points $x_i \in M$ and radii $\delta_i \in (0, \varepsilon)$ and number $\lambda_i \in (0, \delta_i)$ $(i \in J, J \triangleq (i = 1, \ldots, n))$ exist such that

$$M \subset \{\cup\Omega_{\lambda_i}(x_i) : i \in J\} \tag{2.5}$$

and any solution $x(t)$ possesses the following property: if

$$(x(0) \in \Omega_{\lambda_i}(x_i) \cap M^{\varepsilon_0}) \wedge (\forall t \in [0, t^*), \quad x(t) \in M^{\varepsilon_0}) \tag{2.6}$$

for a certain subscript $i \in J$ and interval $[0, t^*)$ (finite or infinite), then

$$x(t) \in \{\cup\Omega_{\delta_i}(x_i), \ i \in J\}, \quad \forall t \in [0, t^*) \tag{2.7}$$

Proof. Suppose $\varepsilon > 0$ is arbitrary. Taking into account the compactness of the set M, we cover it with a finite number of spheres $S_{\delta_i/2}(x_i)$ $(i \in J)$, where $\delta_i \in (0, \varepsilon)$ are the radii of Γ-sectors with centres at the points $x_i \in M$. Since the function is locally bounded, then, without loss of generality, we can assume that $\| f(x) \| \leq L$ for all $x \in \{\cup S_{\delta i}(x_i) : i \in J\}$, where $L > 0$ is a certain constant. We put $\eta = \min\{\delta_i/4 : i \in J\}$ and for arbitrary $\beta_i \in (0, \delta_i/4)$, $\tau \in (0, \eta)$ we denote by $\beta_{0i} \in (0, \beta_i)$, $\tau_{0i} \in (0, \tau)$ numbers the existence of which for each point x_i is established in Lemma 2.1. Since the sets $\Gamma^{\beta_{0i}}(x_i)$ are open, the set

$$H_1 \triangleq \{\cap\Gamma^{\beta_{0i}}(x_i) : i \in J\} \tag{2.8}$$

is an open neighbourhood of the set M. We put

$$H_2^i \triangleq H_1 \cap S_{\delta_i/2}(x_i), \quad H_2 \triangleq \{\cup H_2^i : i \in J\} \tag{2.9}$$

Then H_2 is an open neighbourhood of the set M. Since M is compact, and H_2 is open, a $\varepsilon_0 \in (0, \varepsilon)$ exists such that

$$M^{\varepsilon_0} \subset H_2 \tag{2.10}$$

We put $\lambda_i = 3\delta_i/4$ $(i \in J)$. Suppose that, for the solution $x(t)$, condition (2.6) is satisfied for a certain index $i_1 \in J$ and an interval $[0, t^*)$. For all $t \in [0, t^*)$ the following inclusion is satisfied

$$x(t) \in \Omega_{\delta_{i_1}}(x_{i_1}) \tag{2.11}$$

and inclusion (2.7) is satisfied, and the lemma is proved.

We will assume that (2.1) is not satisfied for a certain $t \in [0, t^*)$. Since $x(0) \in S_{\lambda_{i_1}}(x_{i_1})$ and the set $\Omega_{\delta_{i_1}}(x_{i_1})$ is a Γ-sector, the greatest point t_1 and the least point t_2, $t_1 < t_2$ from the interval $[0, t^*)$ are obtained, for which

$$\| x(t_1) - x_{i_1} \| = \lambda_{i_1}, \quad \| x(t_2) - x_{i_1} \| = \delta_{i_1} \tag{2.12}$$

Here, for all $t \in [t_1, t_2)$ inclusion (2.11) is satisfied and $x(t) \in M^{\varepsilon_0}$. It follows directly from (2.9) and (2.10) that

$$M^{\varepsilon_0} \subset H_1, \quad M^{\varepsilon_0} \subset \{\cap S_{\delta_i/2}(x_i) : i \in J\} \tag{2.13}$$

Hence, we obtain the subscript $i_2 \in J$ such that

$$x(t_1) \in S_{\delta_{i_2}/2}(x_{i_2}) \cap H_1 \tag{2.14}$$

Since $0 < \beta_{i2} < \delta_{i2}/4$, we obtain from (2.14) and the definition of the set H_1

$$x(t_1) \in S_{\delta_{i_2} - \beta_{i_2}} \cap \Gamma^{\beta_{0i2}}(x_{i_2}) \tag{2.15}$$

It follows from (2.12) that $\|x(t_1) - x(t_2)\| \geqslant \delta_{i1}/4$. Hence, the choice of the numbers τ and τ_{oi} ensures that the inequality $t_2 - t_1 > \tau > \tau_{0i_2}$ is satisfied. Now from (2.15) and Lemma 2.1 we obtain $x(t_1 + \tau_{0i_2}) \in \Gamma(x_{i_2}) \cap S_{\beta_{i2}}(x(t_1))$. Consequently, if we take (2.14) into account we have $x(t_1 + \tau_{0i_2}) \in \Omega^0_{\lambda i_2}(x_{i_2})$. Now, again, if for all $t \in [t_1 + \tau_{0i_2}, t^*)$

$$x(t) \in \Omega_{\delta_{i_2}}(x_{i_2}) \tag{2.16}$$

then the lemma is proved.

If this is not so, then taking the point $t_1 + \tau_{0i_2}$ as the initial point, we obtain the greatest point t_3 and the least point t_4 from the interval $[0, t^*)$, $t_3 < t_4$, such that

$$\| x(t_3) - x_{i_2} \| = \lambda_{i_2}, \quad \| x(t_4) - x_{i_2} \| = \delta_{i_2} \tag{2.17}$$

Here $t_3 < t_1$ and inclusion (2.16) is satisfied for all $t \in [t_3 < t_4)$. It follows from (2.14) and (2.17) that

$$\| x(t_1) - x(t_3) \| \geqslant \delta_{i_2}/4 \tag{2.18}$$

Continuing the above steps, we obtain the sequence of points $t_k \in [0, t^*)$ and the subscripts $i_k \in J$ $(k = 1, 2, \ldots)$ such that

$$x(t) \in \Omega_{\delta_{i_k}}(x_{i_k}) \forall t \in [t_{2k-1}, t_{2k}) \tag{2.19}$$

and, like (2.18), $\| x(t_{2k-1}) - x(t_{2k+1}) \| \geqslant \delta_{i_{k+1}}/4$. The last inequality and the choice of the numbers η and τ ensures that $t_{2k-1} - t_{2k+1} \geqslant \eta > \tau$. Hence, if $t^* < +\infty$, this sequence is finite and terminates so that, as described above, the theorem is proved in this case also. (This situation may correspond to the departures of the trajectory from the set M^{ε_0}.) If $t^* = +\infty$, then $t_k \to +\infty$, and hence for any $t \in [0, +\infty)$ we obtain a number k such that inclusion (2.19) and, of course, (2.7) also, are satisfied.

Lemma 2.3. For any $\tau > 0$ and for a finite covering of the set M by the Γ-sectors $\Omega_{\lambda_i}(x_i)(i \in I \triangleq (1, \ldots, m))$, a $\tau_0 \in (0, \tau)$ and $\delta_0 > 0$ exist such that any solution $x(t)$ possesses the property

$$x(0) \in M^{\delta_0} \Rightarrow x(\tau_0) \in \{\cup \Omega_{\lambda_i}(x_i) : i \in I\} \tag{2.20}$$

Proof. Suppose $\tau > 0$ is arbitrary and the Γ-sectors $\Omega_{\lambda_i}(x_i)$ form a covering of the set M, i.e. $M \subset \{\cup \Omega_{\lambda_i}(x_i): i \in I\}$. Then, obviously, the set of open spheres $S_{\lambda_i}(x_i)(i \in I)$ is also a covering of M. Taking into account the compactness of the set M, we can choose a number $\beta > 0$ so small that the following conditions are satisfied

$$\beta < \min\{\lambda_i : i \in I\}, \quad M^\beta \subset \{\cup S_{\lambda_i - \beta}(x_i) : i \in I\}$$

For each Γ-sector $\Omega_{\lambda_i}(x_i)$ and each of the numbers τ and β, by Lemma 2.1 numbers $\tau_{0i} \in (0, \tau)$ and $\beta_{0i} \in (0, \beta)$ exist such that property (2.1) is satisfied for $x = x_i$, $\delta = \lambda_i$, $\tau_0 = \tau_{0i}$, $\beta_0 = \beta_{0i}$.

We put $\delta_0 = \min\{\beta_{0i} : i \in I\}$ and suppose $x(t)$ is a solution such that $x(0) \in M^{\delta_0}$. Then $x(0) \in \Gamma^{\beta_{0i}}(x_i)$ for all $i \in I$ and $x(0) \in S_{\lambda_i - \beta}(x_i)$ for a certain $i \in I$. Hence, it follows from (2.1) that $x(\tau_0) \in \Gamma(x_i) \cap S_\beta(x(0)) \subset \Gamma(x_i) \cap S_{\lambda_i}(x_i)$ for a certain $i \in I$. Property (2.20) has thereby been established and the lemma is proved.

3. THEOREM ON STABILITY

For a real-value function V, defined in a certain neighbourhood of the set M and numbers γ, we will put $E(V < \gamma) \triangleq \{x: V(x) < \gamma\}$. We define the set $E(V = \gamma)$ similarly. We denote by $D^{*+}V(x)$ the upper right derivative of the function V, by virtue of Eq. (1.4).

Theorem 3.1. Suppose that in a certain neighbourhood M^ρ, $\rho > 0$ of the set M a non-negative locally Lipschitz function $V(x)$ is defined with the following properties:

1. $V(x) = 0 \Leftrightarrow x \in x \in M$;
2. for any Γ-sector $\Omega_\delta(x)$ with vertex $x \in M$ and radius $\delta < \rho$ the inequality $D^{*+}V(x') \leqslant 0$ is satisfied for all $x' \in \Omega_\delta(x)$.

Then, for any $\varepsilon > 0$ and $\tau > 0$ a $\delta > 0$ and a finite covering of the set M by the Γ-sectors $\Omega_{\delta_i}(x_i)$, $x_i \in M (i \in J)$ exist such that any solution $x(t)$ with initial conditions $x(0) \in M^\delta$ is defined for all $t \geqslant 0$ and satisfies the conditions

$$\forall t \geqslant 0, \ x(t) \in M^\varepsilon, \ \forall t \geqslant \tau, \ x(t) \in \{\bigcup \Omega_{\delta_i}(x_i) : i \in J\} \tag{3.1}$$

Proof. Suppose $\varepsilon \in (0, \rho)$ and $\tau > 0$ are arbitrary. By Lemma 2.2, $\varepsilon_0 \in (0, \varepsilon)$, Γ-sectors $\Omega_{\delta_i}(x_i)$ and numbers $\lambda_i \in (0, \delta_i)(i \in J)$ exist such that the relation (2.6) ⇒ (2.7) is satisfied. Without loss of generality we will assume that the set M^{ε_0} belongs to Ω together with its closure, since, by virtue of the compactness of M, this can always be achieved due to the arbitrariness of $\varepsilon > 0$.

For all $\eta \in (0, \varepsilon_0)$ a $\gamma > 0$ exists such that

$$E(V < \gamma) \cap M^{\varepsilon_0} \subset M^\eta \tag{3.2}$$

In fact, we will assume the opposite. Then, $\eta \in (0, \varepsilon_0)$, sequences of numbers $\gamma_i \to +0$ and points $x_i \in E(V < \gamma_i) \cap M^{\varepsilon_0}$ exist such that $x_i \notin M^\eta$. Since the set M^{ε_0} is bounded, we can separate a converging subsequence from the sequence $\{x_i\}$. Without loss of generality we will assume that $x_i \to x_0$. By virtue of the continuity of the function V, we obtain $V(x_0) = 0$ and then $x_0 \in M$. But, on the other hand, $x_i \notin M^\eta$ and therefore $x_0 \notin M$. The contradiction obtained proves relation (3.2).

We put the number $\eta = \varepsilon_0/2$ and for the γ corresponding to it we put $W = E(V < \gamma) \cap M^{\varepsilon_0}$.

It follows from (2.5) that the Γ-sectors $\Omega_{\delta i}(x_i)$ form a covering of the set M. Then, by Lemma 2.3, $\tau_0 \in (0, \tau)$ and $\delta_0 > 0$ exist such that condition (2.20) is satisfied. Taking into account the compactness of the set M, the local boundedness of the function f and the arbitrariness of $\tau > 0$, we conclude that the numbers δ_0 and τ_0 can be so small that the inclusion $M^{\delta_0} \subset W$ is satisfied, and any solution $x(t)$ with initial condition $x(0) \in M^{\delta_0}$ is defined and the condition $x(t) \in W$ is satisfied for all $t \in [0, \tau_0]$. Then, for a certain subscript i

$$x(\tau_0) \in W \cap \Omega_{\lambda_i}(x_i) \tag{3.3}$$

If $x(t) \in W$ for all $t \in [0, \omega)$, then, by the theorem on the continuity of solutions [6] and conditions (3.3), (2.6) and (2.7) we have $\omega = +\infty$, and inclusions (3.1) hold, and hence the theorem is proved.

We will assume that a $t \in [0, \omega)$ exists such that $x(t) \notin W$. We then obtain the least value of $t_1 \in [0, \omega)$ such that $x_1 = x(t_1) \in \partial W$. It follows from the definition of the set W and the condition $W \subset M^{\varepsilon_0/2}$ that $x_1 \in E(V = \gamma)$, and then, from the properties of the function V, we obtain

$$\forall t \in [0, t_1), \ V(x(t)) \leqslant V(x_1) = \gamma \tag{3.4}$$

But $x(t) \in W \subset E(V < \gamma)$ for all $t \in [0, t_1)$ and hence $V(x(t)) < \gamma$ when $t \in [0, t_1)$, which contradicts (3.4). The contradiction obtained completes the proof of the theorem.

Note that in Theorem 3.1 not only in the stability of the set M asserted (i.e. $x(0) \in M^\delta \Rightarrow \forall t \geqslant 0$, $x(t) \in M^\varepsilon$), but also the satisfaction of the second condition (3.1), from which it follows that the function $V(x(t))$ does not increase for all $t \geqslant \tau$ along any solution $x(t)$ with initial condition $x(0) \in M^\delta$. The latter, together with the principle of invariance, can be used to prove the asymptotic stability of the set M.

Theorem 3.2. Suppose the conditions of Theorem 3.1 are satisfied and, in addition, $D^{*+}V(x') < 0$ for all $x' \in \Omega^\delta(x)\backslash M$. Then, the set M is asymptotically stable (i.e. M is stable and $d(x(t), M) \to 0$ as $t \to +\infty$ for any solution $x(t)$ with initial value $x(0) \in M^\delta$).

Proof. For the solution $x(t)$ we will denote the ω-limit sets by $\Lambda^+(x)$. Since the function $V(x(t))$ does not increase for all $t > \tau$, by the results obtained is [4], the following condition is satisfied

$$\Lambda^+(x) \subset E(D^{*+}V = 0) \tag{3.5}$$

Suppose $x_0 \in \Lambda^+(x)\backslash M$. Then a sequence $t_k \to +\infty$ exists such that $x(t_k) \to x_0$. By virtue of the second inclusion of (3.1) we obtain the Γ-sector $\Omega_{\delta i}$ (x_i), so that $x_0 \in \Omega_{\delta_i}(x_i)$. But then $D^{*+}V(x_0) < 0$, which contradicts (3.5). The contradiction obtained shows that $\Lambda^+(x) \subset M$, whence the assertion of the theorem follows.

Theorem 3.3. Suppose the conditions of Theorem 3.1 are satisfied and, in addition, the set $E(D^{*+}V = 0) \cap M^\rho$ does not contain closed semi-invariant sets, which do not intersect with M and which belong to some covering of the set M by the Γ-sectors. Then the set M is asymptotically stable.

Proof. Using condition (3.5), the semi-invariance and closedness of the set $\Lambda^+(x)$ (see [4]) and Theorem 3.1, we conclude that with the above assumptions $\Lambda^+(x) \cap M \neq \varnothing$. Then, a sequence $t_k \to +\infty$ exists such that $x(t_k) \to x_0 \in M$. Then, it follows from the stability of the set M that $d(x(t), M) \to 0$ as $t \to +\infty$, and the theorem is proved.

Theorem 3.4. Suppose M is a stable compact set of equilibrium positions of Eq. (1.3), such that some of its neighbourhood M^ρ contains no equilibrium positions which do not belong to M. We will assume that on the set M^ρ locally Lipschitz functions $V_i(x)$ $(i \in I)$ are defined so that for any Γ-sector $\Omega_\delta(x)$ with vertex $x \in M$ and radius $\delta \in (0, \rho)$ the following conditions are satisfied:

1. $D^{*+}V_i(x') \leqslant 0$ for all $x' \in \Omega_\delta(x)$ and $i \in I$;
2. $\Omega_d(x) \cap E \subset \{x : x = (q, \dot q), \dot q = 0\}$, where $E \triangleq \{\cap E(D^{*+}V_i = 0) : i \in I\}$.

Then M is asymptotically stable.

Proof. It follows from the stability of the set M and Lemmas 2.2 and 2.3 that fairly small numbers $\delta > 0$ and $\tau > 0$ exist such that, for any solution $x(t)$ with initial condition $x(0) \in M^\delta$ for all $t > \tau$, the second condition of (3.1) is satisfied for a certain covering of the set M by the Γ-sectors $\Omega_{\delta_i}(x_i)$. Hence, with the assumptions made, the functions $V_i(x(t))$ do not increase when $t > \tau$. Then, $\Lambda^+(x) \subset E$, and hence, taking (3.2) into account we conclude that for any point $x = (q, \dot q) \in \Lambda^+(x)$ the equation $q = 0$ is satisfied. It now follows from the semi-invariance of the set $\Lambda^+(x)$ that it consists of equilibrium positions. Consequently, $\Lambda^+(x) \subset M$, and hence we obtain the assertion of the theorem.

4. EXAMPLE

Consider a plane mechanical system consisting of a piston B of mass m_1, moving with friction along a horizontal rectilinear tube Ox and regarded as a point mass with coordinates $x = q^1$, and a heavy absolutely rigid body having a mass m_2 and moment of inertia J_C about the centre of mass, rotating with friction around a cylindrical hinge, attached to the piston. The distance from the piston to the centre of mass C is r. The angle of deviation β of BC from the normal to Ox, directed downwards, is taken as q^2. We will assume that forces of elasticity of a spring with coefficient of elasticity c and with a point of unstressed state $x = 0$ acts along Ox. The friction coefficients f_1 in the piston and f_2 in the hinge are assumed to be constant, and $m = m_1 + m_2$, $J = J_C + m_2r^2$. This example is a modified form of the system in [3] (see also [4]): we assume that the elastic force of the spring acts along the Ox axis.

The equations of motion of the system in Lagrangian form are written as follows:

$$\begin{aligned} m\ddot x + m_2 r\cos\beta\ddot\beta &= m_2 r\dot\beta^2 \sin\beta - cx + Q_1^T \\ m_2 r\cos\beta\ddot x + J\ddot\beta &= -m_2 g r\sin\beta + Q_2^T \end{aligned} \tag{4.1}$$

The generalized friction forces are found from (1.2) for $s = 1, 2$, where

$$|N_1| = |m_2 r(\ddot{\beta}\sin\beta + \dot{\beta}^2\cos\beta) + mg|$$

$$|N_2| = m_2[(\ddot{x} + r\ddot{\beta}\cos\beta - r\dot{\beta}^2\sin\beta)^2 + (r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g)^2]^{1/2}$$

$$Q_1^{T0} = m_2 r(\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta) + cx \quad (\dot{x} = 0,\ \ddot{x} = 0)$$

$$Q_2^{T0} = m_2 r(\ddot{x}\cos\beta + g\sin\beta) \quad (\dot{\beta} = 0,\ \ddot{\beta} = 0)$$

The sufficient conditions for Eqs (4.1) to be solvable for $\ddot{q} = (\ddot{x}, \ddot{\beta})$ and for them to be reducible to the form (1.3) are inequalities (6.4) from [3]. These inequalities ensure that Lemma 1.1 holds and also the existence of right-hand sided solutions of Eq. (4.1) and their general properties, which were used in this paper (the continuity and semi-invariance of the ω-limit sets).

The set of equilibrium positions for system (4.1) has the form

$$M = \{(q, \dot{q}) : \dot{x} = 0,\ \dot{\beta} = 0,\ f_1 mg \geqslant c|x|,\ f_2 \geqslant r|\sin\beta|\}$$

We will assume that $f_2/r < 1$. In this case M is a set of rectangles in the (x, β) plane. We will determine β_{lz} from the conditions: $\sin\beta_{lz} = f_2/r$, $0 < \beta_{lz} < \pi/2$ and put $x_{lz} = f_1 mg/c$. The set $M_{lz} \triangleq \{(q, \dot{q}) \in M : |x_{lz}|, |\beta| \leqslant \beta_{lz}\}$ will be called the lower stagnation zone.

We will put

$$W_1 = \begin{cases} c(x^2 - x_{lz}^2)/2, & |x| > x_{lz} \\ 0, & |x| \leqslant x_{lz} \end{cases}, \quad W_2 = \begin{cases} m_2 gr(\cos\beta_{lz} - \cos\beta), & |\beta| > \beta_{lz} \\ 0, & |\beta| \leqslant \beta_{lz} \end{cases}$$

$$T = \tfrac{1}{2}(m\dot{x}^2 + 2m_2 r\dot{x}\dot{\beta}\cos\beta + J\dot{\beta}^2), \quad V = T + W_1 + W_2$$

$$w_1 = f_1|N_1||\dot{x}|, \quad w_2 = f_2|N_2||\dot{\beta}|$$

The function V is positive definite with respect to the sets M_{lz} in a certain fairly small neighbourhood of it.

We will write the set Γ for the points $(x_0, \beta_0, \dot{x}_0, \dot{\beta}_0) = (q_0, \dot{q}_0) \in M_{lz}$ and values of the right derivative D^+V by virtue of system (4.1) on each Γ-sector $\Omega_\delta(q_0, \dot{q}_0)$. We note initially that when $(q_0, \dot{q}_0) \in M_{lz}$ we have

$$\dot{q}_0 = 0, \quad \ddot{q}_0 = 0$$

$$|N_1| = mg, \quad |N_2| = m_2 g, \quad |Q_1^{T0}| = c|x|, \quad |Q_2^{T0}| = m_2 rg|\sin\beta|$$

Taking into account the fact that when $|\beta_0| = \beta_{lz}$ the signs of β and $\sin\beta$ are the same in a fairly small neighbourhood of the point β_0, we will consider the following possible cases (the quantity D^+V corresponds to the points $(x, \dot{x}, \beta, \dot{\beta}) = (q, \dot{q}) \in \Omega_\delta(q_0, \dot{q}_0)$)

1. $|x_0| < x_{lz}$, $|\beta_0| < \beta_{lz}$ ($(q_0, \dot{q}_0)$ is an internal point of the rectangle M_{lz}); then

$$\Gamma(q_0, \dot{q}_0) = \{(q, \dot{q}) : \dot{x} = 0,\ \dot{\beta} = 0\}, \quad D^+V = 0$$

2. $|x_0| < x_{lz}$, $|\beta_0| < \beta_{lz}$ or $|\beta_0| < \beta_{lz}$, $|x_0| < x_{lz}$ (the sides of the rectangle M_{lz}) without the vertices; then

$$\Gamma(q_0, \dot{q}_0) = \{(q, \dot{q}) : \dot{x} = 0,\ \dot{\beta}\beta_0 \leqslant 0\}$$

$$D^+V = \begin{cases} -w_2, & |\beta| > \beta_{lz} \\ -w_2 + m_2 gr|\sin\beta||\dot{\beta}|, & |\beta| \leqslant \beta_{lz} \end{cases}$$

3. or, correspondingly

$$\Gamma(q_0, \dot{q}_0) = \{(q, \dot{q}) : \dot{\beta} = 0,\ \dot{x}x_0 \leqslant 0\}$$

$$D^+V = \begin{cases} -w_1, & |x| > x_{lz} \\ -w_1 + c|x||\dot{x}|, & |x| \leqslant x_{lz} \end{cases}$$

4. $|x_0| = x_{lz}$, $|\beta_0| = \beta_{lz}$ (the vertices of the rectangle M_{lz}); then

$$\Gamma(q_0, \dot{q}_0) = \{(q, \dot{q}) : \dot{\beta}\beta_0 \leqslant 0,\ \dot{x}x_0 \leqslant 0\}$$

$$D^+V = \begin{cases} -w_1 - w_2, & |\beta| > \beta_{lz},\ |x| > x_{lz} \\ -w_1 - w_2 + c|x||\dot{x}|, & |x| \leqslant x_{lz},\ |\beta| > \beta_{lz} \\ -w_1 - w_2 + m_2 gr|\sin\beta||\dot{\beta}|, & |\beta| \leqslant \beta_{lz},\ |x| > x_{lz} \\ -w_1 - w_2 + c|x||\dot{x}| + m_2 gr|\sin\beta||\dot{\beta}|, & |\beta| \leqslant \beta_{lz},\ |x| \leqslant x_{lz} \end{cases}$$

Hence, we have nine possible forms for the sets Γ, and within each Γ-sector the generalized velocities $\dot{x}$, $\dot{\beta}$ either vanish or retain signs opposite to the sign of x_0 and β_0, respectively.

In case 1 the sign definiteness of $D^+V(q, \dot{q})$ will not be needed in the further analysis. In cases 2 and 3 the sign of D^+V is determined by the ratios of the values of the functions $f_1|N_1|$ and $c|x|$, $f_2|N_2|$ and $m_2gr|\sin\beta|$ on the set $\Gamma(q, \dot{q})$. It is easy to see that the condition $D^+V \leqslant 0$ will be satisfied (within the Γ-sector), if, for any point $(q_0, \dot{q}_0) \in M_{lz}$ along each solution of Eq. (4.1) with values in the Γ-sector $\Omega_\delta(q_0, \dot{q}_0)$, the following inequality is satisfied

$$D^+\dot{\beta}\sin\beta + \dot{\beta}^2\cos\beta \geqslant 0 \tag{4.2}$$

In fact, in this case it follows from the inequality $|x| \leqslant x_{lz}$ that

$$f_1|N_1| \geqslant f_1 mg \geqslant c|x|$$

and it follows from the inequality $|\beta| \leqslant \beta_{lz}$ that

$$f_2|N_2| \geqslant f_2 m_2 g \geqslant m_2 g r|\sin\beta|$$

whence, taking into account the form of D^+V we also obtain $D^+V \leqslant 0$.

In order to prove (4.2) we will assume the opposite. Then, since the function $D^+\dot{\beta}(t)$ is continuous from the right [6, Theorem 3], the following inequality is satisfied

$$D^+\dot{\beta}\sin\beta + \dot{\beta}^2\cos\beta < 0 \tag{4.3}$$

in a certain small interval $[0, \alpha)$. Integrating (4.3) we obtain

$$\dot{\beta}(t)\sin\beta(t) - \dot{\beta}(0)\sin\beta(0) < 0 \quad \forall t \in (0, \alpha)$$

If $|\beta_0| < \beta_{lz}$, we have $\dot{\beta}(t) = 0$ for sufficiently small $t > 0$, and, consequently, inequality (4.2) is satisfied. Hence, inequality (4.3) will only be satisfied when $|\beta_0| = \beta_{lz}$. We can then assume that $\sin\beta(t) \neq 0$.

Suppose, to fix our ideas, that $\sin\beta(t) > 0$. Then $\dot{\beta}(t) \leqslant 0$ and, consequently, $\sin\beta(t)$ is a non-increasing function. Hence, the inequality $\sin\beta(t) \leqslant \sin\beta(0)$ is satisfied, from which, if we take into account that $\dot{\beta}(t) \leqslant 0$, we obtain the inequality

$$\dot{\beta}(0)\sin\beta(t) \geqslant \dot{\beta}(0)\sin\beta(0) \geqslant \dot{\beta}(t)\sin\beta(t)$$

Consequently, $\dot{\beta}(0) \geqslant \dot{\beta}(t)$, whence we obtain $D^+\dot{\beta}(0) \geqslant 0$. But then inequality (4.3) is not satisfied when $t = 0$, which contradicts the above assumption.

The case $\sin\beta(t) < 0$ is considered in exactly the same way and also leads to a contradiction with (4.3).

Hence, for Eqs (4.1), the sets M_{1z} and the functions V satisfy all the conditions of Theorem 3.1, according to which the lower stagnation zone is stable.

In order to investigate the asymptotic stability of the set M_{1z}, using Theorem 3.4 we consider the functions $V_1 = x^2/2$, $V_2 = \beta^2$. Then, for any $(q_0, \dot{q}_0) \in M_{lz}$ and Γ-sectors $\Omega_\delta(q_0, \dot{q}_0)$ we have

$$D^+V_1 = x\dot{x} \leqslant 0, \quad D^+V_2 = \beta\dot{\beta} \leqslant 0$$

If $x_0 = 0$ for $\beta_0 = 0$, then for sufficiently small $\delta > 0$ for all $(q, \dot{q}) \in \Omega_\delta(q_0, \dot{q}_0)$ we have $\dot{x} = 0$, or, correspondingly, $\dot{\beta} = 0$. If $x_0 \neq 0$ and $\beta_0 \neq 0$, we have

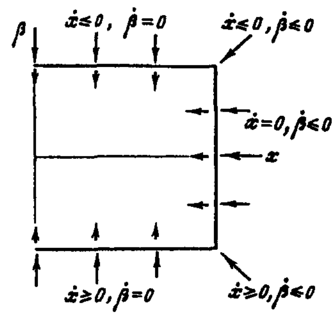

Fig. 1.

$$E(D^+V_1 = 0) \cap E(D^+V_2 = 0) = \{(q, \dot{q}) : \dot{x} = 0, \ \dot{\beta} = 0\}$$

Hence, always

$$\Omega_\delta(q_0, \dot{q}_0) \cap E(D^+V_1 = 0) \cap E(D^+V_2 = 0) \subset \{(q, \dot{q}) : \dot{x} = 0, \ \dot{\beta} = 0\}$$

and, by Theorem 3.4, the set M_{1_z} is asymptotically stable.

In conclusion we note that the second condition of (3.1) of Theorem 3.1 and the relation (2.6) $\Rightarrow$ (2.7) of Lemma 2.2 enable us to give clear geometrical interpretations of the behaviour of the motions in the region of the set M of equilibrium positions (both stable and unstable), since the behaviour of the trajectories within the Γ-sector can be simplified considerably. For system (4.1) the four-dimensional space of the variables (x, β, $\dot{x}$, $\dot{\beta}$) is a phase space. Figure 1 gives a fairly complete representation of the behaviour of the trajectories in the region of the lower stagnation zone (we only show the right-hand part of the (x, β) plane, in the left-hand part the set M_{1z} and the trajectories are symmetrical about the Ox axis). For the Γ-sectors with vertices inside the rectangle M_{1z} only steady motions are possible.

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